

ERGODIC MAXIMIZING MEASURES OF NON-GENERIC, YET DENSE CONTINUOUS FUNCTIONS

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ABSTRACT. Ergodic optimization aims to single out dynamically invariant Borel probability measures which maximize the integral of a given “performance” function. For a continuous self-map of a compact metric space and a dense set of continuous performance functions, we show that the existence of uncountably many ergodic maximizing measures. We also show that, for a topologically mixing subshift of finite type and a dense set of continuous functions there exist uncountably many ergodic maximizing measures which are fully supported and have positive entropy.

1. INTRODUCTION

Ergodic optimization aims to single out dynamically invariant Borel probability measures which maximize the integral of a given “performance” function. It originates in variational methods in mechanical systems [Man, Mat] and several applications can be considered, for instance, in the thermodynamic formalism [BLL], in chaos control [HO, YH]. (See bibliographical notes in [J] for more details).

Consider a continuous self-map T of a compact metric space X and a continuous performance function $\phi : X \rightarrow \mathbb{R}$. A T -invariant Borel probability measure μ is called a ϕ -*maximizing measure* if

$$\max_{\nu \in M(X, T)} \int \phi \, d\nu = \int \phi \, d\mu,$$

where $M(X, T)$ denotes the space of T -invariant Borel probability measures endowed with the weak*-topology. We investigate properties of ϕ -maximizing measures by considering the set $M_{\max}(\phi)$ of all ϕ -maximizing measures. A performance function ϕ is *uniquely maximized* if $M_{\max}(\phi)$ is a singleton. Jenkinson shows that a generic continuous function is uniquely maximized [J]. If T has the specification property, the unique maximizing measure of a generic continuous function is fully supported and has zero entropy [BJ, Br, J, Mo]. However, it is difficult to tell whether or not a given performance function is generic. Indeed, no concrete example of a continuous function is known which is uniquely maximized by a fully supported measure [J, Problems 3.9 and 4.3]. Hence it is natural to investigate properties which are non-generic, yet hold for a reasonably large set of functions.

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In this paper we pay attention to non-generic properties of maximizing measures. Denote by $C(X)$ the space of continuous functions endowed with the supremum norm and by $M_e(X, T)$ the set of ergodic elements of $M(X, T)$. Recall that $M_e(X, T)$ is the set of extrema points of $M(X, T)$. Let us say that $M_e(X, T)$ is arcwise-connected if for every μ and $\nu \in M_e(X, T)$ there exists a homeomorphism $[0, 1] \ni t \mapsto f_t \in M_e(X, T)$ on its image such that $f_0 = \mu$ and $f_1 = \nu$. Note that if $M_e(X, T)$ is arcwise-connected then $M(X, T)$ is not a singleton. We prove that the set of uncountably maximized continuous functions is dense in $C(X)$, provided $M_e(X, T)$ is arcwise-connected.

Theorem A. *Let T be a continuous self-map of a compact metric space X . Suppose $M_e(X, T)$ is arcwise-connected. There exists a dense subset \mathcal{D} of $C(X)$ such that for every ϕ in \mathcal{D} the set $M_{\max}(\phi)$ contains uncountably many ergodic elements.*

Examples to which Theorem A applies include topologically mixing subshifts of finite type and Axiom A diffeomorphisms. The space of invariant measures of these systems is the Poulsen simplex [Si1]: the infinite simplex for which the set of its extremal points is dense. The denseness of the set of ergodic measures implies its arcwise-connectedness because the Poulsen simplex and the set of its extremal points are homeomorphic to the Hilbert cube $[0, 1]^\infty$ and its interior $(0, 1)^\infty$ respectively [GK]. Sigmund shows that the specification, which the above examples actually have, implies the denseness of the set of ergodic elements [Si2]. Several extensions of Sigmund's result have been considered under some generalized versions of the specification (see [GK]). In one-dimensional case, Blokh shows that continuous topologically mixing interval maps have the specification [Bl] and a discontinuous version is studied in [Bu].

The arcwise-connectedness of the set of ergodic measures is strictly weaker than the denseness of it. For example the set of ergodic measures of the Dyck shift [Kr] is not dense but arcwise-connected. The connectedness of the set of ergodic measures for some partially hyperbolic systems is studied in [GP]. On the other hand, there do exist systems for which $M_e(X, T)$ is not arcwise-connected: Cortez and Rivera-Letelier show that for the restriction of some logistic maps to the omega limit set of the critical points, the sets of ergodic measures become totally disconnected [CR].

An idea of our proof of Theorem A is to perturb a given continuous function ϕ_0 to create another ϕ so that the function $\mu \mapsto \int \phi d\mu$ defined on an arc of ergodic measures has a “flat” part (see FIGURE 1). The Bishop Phelps theorem allows us to construct such a perturbation. In order to use the Bishop Phelps theorem, we use the fact that maximizing measures are characterized as “tangent measures” to the convex functional

$$Q : C(X) \ni \phi \mapsto \max_{\nu \in M(X, T)} \int \phi d\nu \in \mathbb{R}.$$

See Proposition 3. The use of the Bishop Phelps theorem has been inspired by [PU] (see also [I]).

It is worthwhile to remark that our perturbation does not work in the Lipschitz topology. In the course of the proof of Theorem A we show the following statement.

Corollary. *Let $\phi_0 \in C(X)$ and $\mu \in M_{\max}(\phi_0)$. For any neighborhood U of ϕ_0 and any open neighborhood V of μ there exists ϕ in U such that $V \cap M_{\max}(\phi)$ contains uncountably many ergodic elements.*

On the other hand, in the space of Lipschitz continuous functions a phenomenon called a “lock up on periodic orbits” occurs: for a Lipschitz continuous functions which is uniquely maximized by a periodic measure one cannot realize a perturbation breaking the uniqueness of maximizing measure [YH] (See also [BLL]).

For the subshift of finite type, one can choose the arc of ergodic measures used in the proof of Theorem A from fully supported measures with positive entropy. Hence, slightly modifying the proof of Theorem A we obtain the next theorem.

Theorem B. *Let (X, T) be a topologically mixing subshift of finite type. There exists a dense subset \mathcal{D} of $C(X)$ such that for every ϕ in \mathcal{D} the set $M_{\max}(\phi)$ contains uncountably many ergodic elements which are fully supported and positive entropy.*

The rest of this paper is organized as follows. In Section 2 we collect preliminary results on functional analysis and invariant Borel probability measures. In Section 3 the theorems are proved.

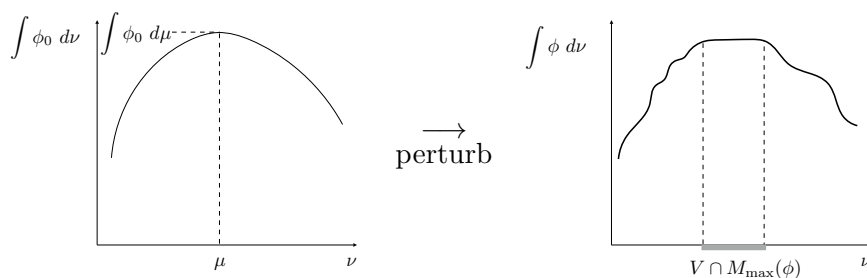


FIGURE 1. A schematic picture of the perturbation: a given function ϕ_0 (left); the perturbed one ϕ (right).

2. PRELIMINARIES

2.1. Bishop Phelps Theorem. First we see the Bishop Phelps theorem, which is concerned with a convex functional on a Banach space. We begin with definitions of basic notions.

Definition 1. A functional $\Gamma : V \rightarrow \mathbb{R}$ on a Banach space V is *convex* if

$$\Gamma(t\phi + (1-t)\psi) \leq t\Gamma(\phi) + (1-t)\Gamma(\psi)$$

for all $\phi, \psi \in V$ and $t \in [0, 1]$.

Let $\Gamma : V \rightarrow \mathbb{R}$ be a convex and continuous functional on a Banach space V . A bounded linear functional F is *tangent* to Γ at $\phi \in V$ if

$$F(\psi) \leq \Gamma(\phi + \psi) - \Gamma(\phi)$$

for all $\psi \in V$.

A bounded linear functional F is *bounded* by Γ if $F(\psi) \leq \Gamma(\psi)$ for all $\psi \in V$.

For a bounded linear functional F on a Banach space V define

$$\|F\| = \sup\{|F(\phi)| : \phi \in V \text{ with } \|\phi\| = 1\}.$$

This becomes a norm of the set V^* of all bounded linear functionals on V . Note that V^* becomes a Banach space with this norm. The Bishop Phelps theorem states that Γ -bounded functionals can be approximated by Γ -tangent ones with respect to this norm.

Theorem 2. [I, Theorem V.1.1.] *Let $\Gamma : V \rightarrow \mathbb{R}$ be a convex and continuous functional on a Banach space V . For every bounded linear functional F_0 bounded by Γ , $\phi_0 \in V$ and $\varepsilon > 0$, there exist a bounded linear functional F and $\phi \in V$ such that F is tangent to Γ at ϕ and*

$$\|F_0 - F\| \leq \varepsilon \quad \text{and} \quad \|\phi_0 - \phi\| \leq \frac{1}{\varepsilon}(\Gamma(\phi_0) - F_0(\phi_0) + s(F_0)),$$

where $s(F_0) = \sup\{F_0(\psi) - \Gamma(\psi) : \psi \in V\}$.

2.2. The Space of Borel Probability Measures. Denote by $M(Y)$ the set of all Borel probability measures on a compact metric space Y . In our setting, $M(X, T)$ is compact and metrizable in the weak*-topology. Then we will also consider $M(M(X, T))$ in the following section. Denote by $C(Y)$ the Banach space of all continuous functions on Y with the supremum norm. By the Riesz representation theorem, the set

$$\{F \in C(Y)^* : F \text{ is positive and normalized}\}$$

can be identified with $M(Y)$. With this identification, the norm and the notion of being tangent and being bounded in Definition 1 carry over to elements of $M(Y)$. For $\mu \in M(X, T)$ we denote by $\mu(\phi)$ the integral of a continuous function ϕ by μ .

2.3. Ergodic Decomposition. For a T -invariant measure μ there exists a unique Borel probability measure α on $M(X, T)$ such that $\alpha(M(X, T) \setminus M_e(X, T)) = 0$ and

$$\mu(\phi) = \int_{M_e(X, T)} m(\phi) d\alpha(m)$$

for all $\phi \in C(X)$. We call α the ergodic decomposition of μ . Note that $M(X, T)$ is a nonempty compact convex set and $M_e(X, T)$ coincides with the set of its extremal points. Since the ergodic decomposition of a T -invariant measure is unique, $M(X, T)$ is a Choquet simplex. From the Theory of a Choquet simplex, for the ergodic decompositions α_1, α_2 of $\mu_1, \mu_2 \in M(X, T)$, we have $\|\mu - \nu\| = \|\alpha_\mu - \alpha_\nu\|$. See [R, Appendix A.5] and the references therein.

3. PROOFS OF THE THEOREMS

3.1. On the proof of Theorem A. Define a functional $Q : C(X) \rightarrow \mathbb{R}$ by

$$Q(\phi) = \max \{\mu(\phi) : \mu \in M(X, T)\}.$$

Note that Q is continuous and convex. Maximizing measures are characterized by tangency to Q .

Proposition 3. [Br, Lemma 2.3] *Let T be a continuous self-map of a compact metric space X and $\phi \in C(X)$. Then $\mu \in M(X, T)$ is tangent to Q at ϕ if and only if $\mu \in M_{\max}(\phi)$.*

First we consider the ergodic decomposition of a ϕ -maximizing measure. The following proposition states that invariant measures in the support of the ergodic decomposition of a ϕ -maximizing measure are also ϕ -maximizing. The support of an ergodic decomposition α is defined by $\text{supp}(\alpha) = \bigcap C$ where the intersection is taken over all closed subsets C of $M(X, T)$ with $\alpha(C) = 1$. Note that $\alpha(\text{supp}(\alpha)) = 1$, since $M(X, T)$ has a countable basis.

Proposition 4. *Let T be a continuous self-map of a compact metric space X and $\phi \in C(X)$. Let $\mu \in M_{\max}(\phi)$ and let α be the ergodic decomposition of μ . Then $\text{supp}(\alpha)$ is contained in $M_{\max}(\phi)$.*

Proof. Let $N = \{\nu \in M(X, T) : \int \phi d\nu < \int \phi d\mu\}$. Suppose $\alpha(N) > 0$. Then

$$\begin{aligned} \mu(\phi) &= \int_{M(X, T)} m(\phi) d\alpha(m) \\ &= \int_N m(\phi) d\alpha(m) + \int_{M(X, T) \setminus N} m(\phi) d\alpha(m) \\ &< \alpha(N)\mu(\phi) + \alpha(M(X, T) \setminus N)\mu(\phi) \\ &= \mu(\phi). \end{aligned}$$

This is a contradiction and then we have $\alpha(N) = 0$. Since $M(X, T) \setminus N = M_{\max}(\phi)$ and $M_{\max}(\phi)$ is closed, we have $\text{supp}(\alpha) \subset M_{\max}(\phi)$. \square

Second we construct a non-atomic Borel probability measure α on $M(X, T)$ supported in $M_e(X, T)$ for a given $\phi \in C(X)$. The point of the construction is this $\text{supp}(\alpha)$ gives positive weight to the set of ergodic measures for which the integrals of ϕ are ε -close to the maximum value $Q(\phi)$. The arcwise-connectedness of $M_e(X, T)$ is essential for the following construction.

Proposition 5. *Let T be a continuous self-map of a compact metric space X . Suppose $M_e(X, T)$ is arcwise-connected. Then for every $\phi \in C(X)$ there is a non-atomic Borel probability measure α on $M(X, T)$ such that $\text{supp}(\alpha) \subset M_e(X, T)$ and for all $\varepsilon > 0$,*

$$\alpha(\{\mu \in M_e(X, T) : Q(\phi) - \varepsilon \leq \mu(\phi)\}) > 0.$$

Proof. Pick $\phi \in C(X)$ and let μ be a ϕ -maximizing measure. Pick $\nu \in M_e(X, T) \setminus \{\mu\}$. By the assumption, there exists an arc f from μ to ν . Let $\text{Leb}_{[0, 1]}$ denote the Lebesgue measure on $[0, 1]$ and $\alpha = f_*\text{Leb}_{[0, 1]}$. Since f is a homeomorphism, the inverse image of a point in $f([0, 1])$ is a singleton. Hence α is non-atomic. Since $t \in [0, 1] \mapsto \int \phi df_t$ is continuous, the set $\{t \in [0, 1] : Q(\phi) - \varepsilon \leq \int \phi df_t\}$ has nonempty interior. Since the image of the set by f is contained in $M_e(X, T)$, α is supported in $M_e(X, T)$ and satisfies the desired inequality. \square

We now prove Theorem A.

Proof of Theorem A. Note that every Borel probability measure μ is bounded by Q . Pick $\phi_0 \in C(X)$ and $0 < \varepsilon < \frac{1}{2}$. Let $\tilde{\alpha}$ be a non-atomic Borel probability measure for which the conclusion of Proposition 5 holds with $\phi = \phi_0$. Put $\delta = \varepsilon^2$ and

$$A_\delta = \{\mu \in M_e(X, T) : Q(\phi_0) - \delta \leq \mu(\phi_0)\}.$$

Denote by α_0 the conditinal measure of α_0 on A_δ , namely

$$\alpha_0(B) = \frac{1}{\tilde{\alpha}(A_\delta)} \tilde{\alpha}(A_\delta \cap B)$$

for every Borel subsets B of $M(X, T)$. Since $\tilde{\alpha}(A_\delta) > 0$ by Proposition 5, α_0 is well-defined. Note that α_0 is also supported in $M_e(X, T)$.

Let $\mu_0 = \int_{M_e(X,T)} m \, d\alpha_0(m)$. By Theorem 2 applied to ϕ_0 , μ_0 and ε , there exist $\phi \in C(X)$ and $\mu \in M(X)$ such that μ is tangent to Q at ϕ , $\|\mu - \mu_0\| \leq \varepsilon$ and

$$\|\phi - \phi_0\| \leq \frac{1}{\varepsilon} (Q(\phi_0) - \mu_0(\phi_0)) \leq \frac{1}{\varepsilon} \delta = \varepsilon.$$

Then ϕ is ε -close to ϕ_0 and by Proposition 3 μ is a ϕ -maximizing measure.

Next we show the existence of uncountably many ergodic ϕ -maximizing measures. Let α be the ergodic decomposition of μ and we have

$$(1) \quad \|\alpha - \alpha_0\| = \|\mu - \mu_0\| \leq \varepsilon.$$

Let $\rho > \frac{1-\varepsilon}{2} > 0$. Since α_0 is a Borel probability measure and $\text{supp}(\alpha)$ is a closed set, there is an open set U such that $\text{supp}(\alpha) \subset U$ and $\alpha_0(U \setminus \text{supp}(\alpha)) < \rho$. Since $M(X, T)$ is a metric space, there is a continuous function $g : M(X, T) \rightarrow [0, 1]$ which vanishes on $M(X, T) \setminus U$ and is identically 1 on $\text{supp}(\alpha)$. Hence we have

$$\alpha_0(\text{supp}(\alpha)) > \alpha_0(U) - \rho \geq \int g \, d\alpha_0 - \rho.$$

The inequality in (1) implies

$$-\varepsilon \leq \int h \, d\alpha - \int h \, d\alpha_0 \leq \varepsilon$$

for all $h \in C(M(X, T))$ with $\|h\| = 1$. Hence we have

$$(2) \quad \begin{aligned} \alpha_0(\text{supp}(\alpha)) &> \int g \, d\alpha_0 - \rho \\ &\geq \int g \, d\alpha - \rho - \varepsilon \\ &\geq \alpha(\text{supp}(\alpha)) - \rho - \varepsilon = 1 - \rho - \varepsilon > 0. \end{aligned}$$

Since α_0 is non-atomic and supported in $M_e(X, T)$, $\text{supp}(\alpha)$ contains uncountably many ergodic elements. By Proposition 4 we have $\text{supp}(\alpha) \subset M_{\max}(\phi)$, and the proof is complete. \square

3.2. On the proof of Theorem B. The following result by Sigmund is essential for our proof of Theorem B. For $\mu, \nu \in M_e(X, T)$ a continuous function $[0, 1] \ni t \mapsto f_t \in M_e(X, T)$ which satisfies $f_0 = \mu$ and $f_1 = \nu$ is called a *path* from μ to ν .

Theorem 6. [Si3] *Let (X, T) be a topologically mixing subshift of finite type. Then for every $\mu, \nu \in M_e(X, T)$ there exists a path f from μ to ν with the following properties: (i) for every measure $m \in f([0, 1])$, $f^{-1}(\{m\})$ is a countable set; (ii) every measure $m \in f([0, 1])$ except for countably many ones is fully supported and has positive entropy.*

Proof of Theorem B. Pick $\phi_0 \in C(X)$ and $0 < \varepsilon < \frac{1}{2}$. We obtain a non-atomic Borel probability measure on $M_e(X, T)$ by modifying the proof of Proposition 5. Let μ be a ϕ_0 -maximizing measure and pick $\nu \in M_e(X, T) \setminus \{\mu\}$. Let f be a path from μ to ν for which the conclusion of Theorem 6 holds. Since the inverse image of any point is countable, $\tilde{\alpha} = f_* \text{Leb}_{[0,1]}$ becomes a non-atomic Borel probability measure supported in $M_e(X, T)$. Then for $\tilde{\alpha}$ the inequality in Proposition 5 holds with ϕ_0 and ε^2 .

Following the proof of Theorem A, we define α_0 to be the restriction of $\tilde{\alpha}$ to the set $A_{\varepsilon^2} = \{\mu \in M_e(X, T) : Q(\phi_0) - \varepsilon^2 \leq \mu(\phi_0)\}$ and obtain $\phi \in C(X)$ and a Borel probability measure α such that $\|\phi_0 - \phi\| \leq \varepsilon$ and $\text{supp}(\alpha) \subset M_{\max}(\phi)$.

By Theorem 6, $\text{supp}(\tilde{\alpha}) = f([0, 1])$ contains uncountably many ergodic elements which are fully supported and have positive entropy. The definition of α_0 implies

$$\text{supp}(\alpha_0) = \text{supp}(\tilde{\alpha}) \cap A_{\varepsilon^2}.$$

and $\text{supp}(\alpha_0)$ still contains uncountably many ergodic elements which are fully supported and have positive entropy. By (2) in the proof of Theorem A we have

$$\alpha_0(\text{supp}(\alpha_0) \cap \text{supp}(\alpha)) = \alpha_0(\text{supp}(\alpha)) > 0.$$

Since α_0 is non-atomic and supported in $M_e(X, T)$, this implies $\text{supp}(\alpha_0) \cap \text{supp}(\alpha)$ contains uncountably many ergodic elements which are fully supported and have positive entropy. By Proposition 4 we have

$$\text{supp}(\alpha_0) \cap \text{supp}(\alpha) \subset \text{supp}(\alpha) \subset M_{\max}(\phi)$$

and the proof is complete. \square

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